A tableau system for modal logic S4 with an efficient proof-search procedure

Toshimasa Matsumoto
tmatsumo@jaist.ac.jp
School of Information Science, JAIST

Abstract

We give a loop-free proof-search procedure for **S4** based on tableau system. The proof-search procedures for **S4** have been studied in [6, 7, 5, 12]. Our tableau system has been inspired by the system in [12]. In our tableau system, redundant repeated applications of tableau rules can be eliminated by a method which is much more efficient than the standard loop-checking. This contributes to technique for implementing of theorem prover for **S4**.

1 Introduction

Tableau system is used to formulate some logics, like classical logic, intuitionistic logic, modal logics, substructural logics [13] and so on [3, 4]. Each tableau systems has a set of tableau rules, by which formulas in a tableau are broken down to simpler formulas. Tableau became widely known by the book [15] of Smullyan. Smullyan used tableaux as the basis of a general treatment of classical logic. Then the method is applied to several modal logics by Fitting [1] and temporal logics by Rescher and Urquhart [14]. In [2], tableau systems for both normal and non-normal modal logics extensively discussed.

In this paper, we will discuss modal tableau systems for **S4**. Especially we consider an efficient proof-search for **S4**. Usually the modal tableau systems for **S4** has the following rules called (T) and $(S4)$:

$$\frac{\Gamma, \Box A}{\Gamma, \Box A, A}\ (T) \qquad\qquad \frac{\Gamma, \Box\Delta, \neg\Box A}{\Box\Delta, \neg A}\ (S4)$$

In this application of (T), the denominator $\Gamma, \Box A, A$ is not simpler than the numerator $\Gamma, \Box A$. Therefore, (T) is applicable once again for this $\Box A$. Moreover, alternating applications of (T) and $(S4)$ may cause redundant repeated applications. It arises from that $\Box\Delta$ in numerator of $(S4)$ remains in its denominator. In order to avoid such redundant repeated applications of (T) and $(S4)$, a simple loop-checking would obviously cause a big inefficiency.

The proof-search procedures for **S4** have been proposed [6, 7, 5, 12]. Strictly speaking, they give proof-search procedures. However, the procedure of [6, 7] holds only for formulas in a certain normal form. That is why we take particular note of Heuerding, Seyfried and Zimmermann's system [5] and Mouri's one [12]. In [5], set to store the necessary information, called *history*, has been introduced. In [12], an auxiliary modal operator [11] and *histories* have been applied to sequent system. The procedure in [12] is not only a proof-search procedure but also a procedure of construction of counter-models.

In our work, by introducing an auxiliary modal operator $\blacksquare$ and *histories* to the ordinary modal tableau system for **S4**, we give a loop-free proof-search procedure for **S4**, which returns a closed tableau as its output if a given formula is a theorem of **S4**, otherwise which tells us that the formula is not a theorem of **S4**. Since the procedure is loop-free, tableaux constructed by it are also loop-free itself, that is, there are no redundant repeated applications of rules. Therefore, our tableau system improves Mouri's system much in its efficiency, because it gives us a loop-free proof-search procedure. Hence, in implementation of the procedure, there is no need for loop-checking. Moreover, by termination of the procedure, we can estimate the time complexity of the procedure.

The present paper consists of five sections. In Section 2, we briefly give the syntax and semantics for modal logics, especially **S4**. In Section 3, a modal tableau system for **S4**, called $\mathcal{C}$**S4+**, is given. Next we show that soundness and completeness of $\mathcal{C}$**S4+**. By proving completeness, we see that $\mathcal{C}$**S4+** is a proof-search procedure. Our completeness proof actually gives a procedure of construction of counter-models for **S4**. Though, a similar result is given by Mouri, our idea is different from Mouri's. Further, we show termination of the $\mathcal{C}$**S4+** and estimate the time complexity of it. In Section 4, we conclude and in the last Section 5, further work is given.

2 Preliminaries

We define the modal logics, especially **S4** and its Kripke semantics. The readers are supposed to be familiar with classical logic.

2.1 Syntax

We assume here that the language $\mathcal{L}$ consists of propositional variables, logical connectives and formulas as defined below.

Let $\mathcal{P}$ be a set of propositional variables say $p, q, r, \cdots$. The symbols $\wedge$, $\vee$, $\supset$, $\neg$ and $\Box$ are called logical connectives. The symbol $\Box$ is a modal operator. Formulas are defined inductively as follows:

Definition 2.1 (formulas)

- *any member of $\mathcal{P}$ is a formula (called an atomic formula),*
- *constant f is a formula,*
- *if A and B are formulas then so are $(A \wedge B), (A \vee B), (A \supset B), (\neg A)$ and $(\Box A)$.*

To omit parentheses, we suppose that $\neg$ binds stronger than any other logical connective. In addition, we can omit the outermost parentheses of formulas. A **literal** is either an atomic formula (positive literal) or negation of an atomic formula (negative literal). A pair $(p, \neg p)$ of literals is said to be complementary. Capital letters $A, B, C, \cdots$ denote arbitrary formulas. Greek capital letters $\Gamma, \Delta, \Pi, \cdots$ denote finite (possibly empty) sets of formulas.

The normal modal logic is defined by adding the the following axiom **K** and the necessitation rule (if A is a theorem then so is $\Box A$) to classical logic. The modal logic **S4** is obtained by adding the following axioms **T** and **4** to the normal modal logic. The modal logic **S4** is also a normal modal logic.

$$\begin{array}{ccl} \mathbf{K} & : & \Box(A \supset B) \supset (\Box A \supset \Box B) \\ \mathbf{T} & : & \Box A \supset A \\ \mathbf{4} & : & \Box A \supset \Box\Box A \end{array}$$

2.2 Semantics

We give a semantics for modal logics. It is given by Kripke frames defined below:

Definition 2.2 (Kripke frames) *A Kripke frame is a pair (W, R) where W is a non-empty set of possible worlds and R, called its accessibility relation, is a binary relation on W. If wRw' then we say that a possible world w' is accessible from a possible world w.*

Definition 2.3 (Kripke models) *Let (W, R) be a Kripke frame and V be a mapping from propositional variables to 2^W, called a valuation on (W, R). A Kripke model is a triple (W, R, V). For a given Kripke model (W, R, V), a relation $\models$ between an element of W and a formula is defined inductively as follows:*

$$\begin{array}{lll} w \models p & \textit{iff} & w \in V(p), \textit{ where } p \in \mathcal{P}, \\ w \models A \wedge B & \textit{iff} & w \models A \textit{ and } w \models B, \\ w \models A \vee B & \textit{iff} & w \models A \textit{ or } w \models B, \\ w \models A \supset B & \textit{iff} & w \models A \textit{ implies } w \models B, \\ w \models \neg A & \textit{iff} & w \not\models A, \\ w \models \Box A & \textit{iff} & \textit{for all } w', wRw' \textit{ implies } w' \models A. \end{array}$$

If $w \models A$ we sometimes also say that w **makes** A **true**. Since the relation $\models$ is uniquely determined by the valuation V, we identify $\models$ with V. Hence, $(W, R, \models)$ can be called a Kripke model.

A formula A is **satisfiable in a model** $(W, R, \models)$ iff there exists some $w \in W$ such that $w \models A$. A formula A is **satisfiable on a frame** (W, R) iff there exists some valuation $\models$ and some world $w \in W$ such that $w \models A$.

Let $\mathbf{L}$ be a modal logic. It is known that particular axioms correspond to particular restrictions on the accessibility relation R. A frame (W, R) is called $\mathbf{L}$-frame if the accessibility relation R meets the particular restriction to which the particular axioms of $\mathbf{L}$ correspond (See [4] for the details). Especially, we say that a frame (W, R) is an **S4-frame** if R is reflexive (corresponding to the axiom **T**) and transitive (corresponding to the axiom **4**).

A Kripke model $(W, R, \models)$ is an **L-model** iff (W, R) is an $\mathbf{L}$-frame, where $\models$ is any valuation. An $\mathbf{L}$-model $(W, R, \models)$ is an **L-model for** Γ iff there exists some $w \in W$ such that $w \models A$ for all $A \in \Gamma$, written down $w \models \Gamma$. A set finite Γ is **L-satisfiable** iff there is an $\mathbf{L}$-model for Γ.

Definition 2.4 *For given some finite set* Γ *and some formula* A*, we say that* A *is a* **logical consequence** *of* Γ *in* $\mathbf{L}$ *iff for every* $\mathbf{L}$*-model* $(W, R, \models)$ *and for every* $w \in W$*, if* $w \models \Gamma$ *then* $w \models A$*. We write* $\Gamma \models_{\mathbf{L}} A$ *if* A *is a logical consequence of* Γ *in* $\mathbf{L}$*.*

Proposition 2.5 $\Gamma \models_{\mathbf{L}} A \iff \Gamma \cup \{\neg A\}$ *is* $\mathbf{L}$*-unsatisfiable.*

Proof) Trivial. $\Box$

3 Modal tableau system for S4

In this section, we give a tableau system for **S4** and show soundness and completeness of it. After that, we can see that our tableau system for **S4** is actually a proof-search procedure for **S4**. Further, we show termination and the time complexity of the procedure.

We write down $\mathcal{C}\mathbf{L}$ for a tableau system for a logic $\mathbf{L}$. A tableau system consists of a set of tableau rules. Each tableau rules consists of a numerator and a list of denominators. The numerator is a finite set of formulas and so is each denominators. A $\mathcal{C}\mathbf{L}$-tableau for a finite set Π is a finite tree with the root Π each of whose nodes carries a finite set of formulas. A tableau is constructed by repeated applications of tableau rules.

3.1 Modal tableau system $\mathcal{C}$S4+

Our modal tableau system for **S4**, called $\mathcal{C}\mathbf{S4}+$, is inspired by Mouri's system in [12]. The tableau rules of $\mathcal{C}\mathbf{S4}+$ are different from the standard tableau rules. Each of numerators and denominators also carries a pair of sets of formulas $\langle \Box\Sigma \mid \neg\Box\Omega \rangle$, where $\Box\Sigma = \{\Box A \mid A \in \Sigma\}$ and $\neg\Box\Omega = \{\neg\Box A \mid A \in \Omega\}$. The pair $\langle \Box\Sigma \mid \neg\Box\Omega \rangle$ is called a *history*. The tableau rules of $\mathcal{C}\mathbf{S4}+$ are given in Figure 1, where we suppose that $\{p_1, \cdots, p_n\}$ and $\{\neg q_1, \cdots, \neg q_m\}$ do not contain complementary pairs each other. The symbol $\blacksquare$ is an auxiliary modal operator which is introduced in [11]. It plays an important role in order to avoid redundant repeated applications of the tableau rule (T) [10]. We call each of $(\wedge)$, $(\vee)$, $(\supset)$, $(\neg\wedge)$, $(\neg\vee)$, $(\neg\supset)$, $(\neg)$ and $(T+)$ a **static rule**, and each of $(S4+)_t$ and $(S4+)_s$ a **transitional rule**. The formula shown explicitly in each static rule is called the **principal formula**.

Now, we give the tableau procedure. When we test whether $\Gamma \models_{\mathbf{S4}} A$ or not, we apply tableau rules to the set with history $(\Gamma \cup \{\neg A\})\langle \epsilon \mid \epsilon \rangle$. As we see, the tableau procedure is a refutation procedure. A tableau rule with numerator N is applicable to a node carrying a set with history $\Delta\langle \Box\Sigma \mid \neg\Box\Omega \rangle$ if $\Delta\langle \Box\Sigma \mid \neg\Box\Omega \rangle$ is an instance of N. The tableau procedure is as follows:

1. input $(\Gamma \cup \{\neg A\})\langle \epsilon \mid \epsilon \rangle$,
2. choose a leaf node n, carrying the set with history $\Delta\langle \Box\Sigma \mid \neg\Box\Omega \rangle$, of current tree,
3. if the rule (f) is applicable to n then apply it to n, and else choose a rule (r) which is applicable to n (Note that the rule (f) has priority over any other rule),
4. if the rule (r) has k denominators $D_1, \cdots, D_k$ then create k successor nodes for n, where the successor i carries an appropriate instance of denominator D_i for each $1 \le i \le k$,
5. otherwise, n is the end node.

$$\frac{\Gamma, p, \neg p \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{f}\ (f) \qquad \frac{\Gamma, A \wedge B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, A, B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (\wedge)$$

$$\frac{\Gamma, A \vee B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle \quad \mid \quad \Gamma, B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (\vee) \qquad \frac{\Gamma, A \supset B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, \neg A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle \quad \mid \quad \Gamma, B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (\supset)$$

$$\frac{\Gamma, \neg(A \wedge B) \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, \neg A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle \quad \mid \quad \Gamma, \neg B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (\neg\wedge) \qquad \frac{\Gamma, \neg(A \vee B) \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, \neg A, \neg B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (\neg\vee)$$

$$\frac{\Gamma, \neg(A \supset B) \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, A, \neg B \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (\neg\supset) \qquad \frac{\Gamma, \neg\neg A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (\neg)$$

$$\frac{\Gamma, \Box A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Gamma, \blacksquare A, A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}\ (T+)$$

$$\frac{p_1, \cdots, p_n, \neg q_1, \cdots, \neg q_m, \neg\Box B_1, \cdots, \neg\Box B_l, \blacksquare\Delta, \neg\Box A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Box\Delta, \neg A \quad \langle \Box\Delta \mid \neg\Box A\rangle}\ (S4+)_t$$
$$\text{if } \Box\Sigma \subsetneq \Box\Delta$$

$$\frac{p_1, \cdots, p_n, \neg q_1, \cdots, \neg q_m, \neg\Box B_1, \cdots, \neg\Box B_l, \blacksquare\Delta, \neg\Box A \quad \langle \Box\Sigma \mid \neg\Box\Omega\rangle}{\Box\Delta, \neg A \quad \langle \Box\Sigma \mid \neg\Box A, \neg\Box\Omega\rangle}\ (S4+)_s$$
$$\text{if } \Box\Sigma = \Box\Delta \text{ and } \neg\Box A \notin \neg\Box\Omega$$

Figure 1: Tableau rules of $\mathcal{C}\mathbf{S4+}$

A $\mathcal{C}\mathbf{S4+}$-tableau is **closed** if all its end nodes are $\{f\}$, otherwise it is **open**. A finite set Π is $\mathcal{C}\mathbf{S4+}$-**consistent** if no $\mathcal{C}\mathbf{S4+}$-tableau for Π is closed, otherwise Π is $\mathcal{C}\mathbf{S4+}$-**inconsistent**. We write down $\Gamma \vdash_{\mathcal{C}\mathbf{S4+}} A$ if $\Gamma \cup \{\neg A\}$ is $\mathcal{C}\mathbf{S4+}$-inconsistent.

We note that same node never appear on each branch from the root to any node in every $\mathcal{C}\mathbf{S4+}$-tableau for any finite set Γ. This is discussed in Section 3.5.

Example 1 The left tableau is a closed $\mathcal{C}\mathbf{S4+}$-tableau for $\neg(\Box p \supset \Box\Box p)$, while the right one is an open $\mathcal{C}\mathbf{S4+}$-tableau for $\neg\neg\Box A$, where A denotes $\neg(\Box p \vee \Box\neg p)$.

$$\begin{array}{cl}
\neg(\Box p \supset \Box\Box p)\ \langle \epsilon \mid \epsilon\rangle & \\ \hline
\Box p, \neg\Box\Box p\ \langle \epsilon \mid \epsilon\rangle & \\ \hline
\blacksquare p, p, \neg\Box\Box p\ \langle \epsilon \mid \epsilon\rangle & \\ \hline
\Box p, \neg\Box p\ \langle \Box p \mid \neg\Box\Box p\rangle & (S4+)_t \\ \hline
\blacksquare p, p, \neg\Box p\ \langle \Box p \mid \neg\Box\Box p\rangle & \\ \hline
\Box p, \neg p\ \langle \Box p \mid \neg\Box p, \neg\Box\Box p\rangle & (S4+)_s \\ \hline
\blacksquare p, p, \neg p\ \langle \Box p \mid \neg\Box p, \neg\Box\Box p\rangle & \\ \hline
f &
\end{array}
\qquad
\begin{array}{cl}
\neg\neg\Box A\ \langle \epsilon \mid \epsilon\rangle & \\ \hline
\Box A\ \langle \epsilon \mid \epsilon\rangle & \\ \hline
\blacksquare A, A\ \langle \epsilon \mid \epsilon\rangle & \\ \hline
\blacksquare A, \neg\Box p, \neg\Box\neg p\ \langle \epsilon \mid \epsilon\rangle & \\ \hline
\Box A, \neg p\ \langle \Box A \mid \neg\Box p\rangle & (S4+)_t \\ \hline
\blacksquare A, A, \neg p\ \langle \Box A \mid \neg\Box p\rangle & \\ \hline
\blacksquare A, \neg\Box p, \neg\Box\neg p, \neg p\ \langle \Box A \mid \neg\Box p\rangle & \\ \hline
\Box A, \neg\neg p\ \langle \Box A \mid \neg\Box\neg p, \neg\Box p\rangle & (S4+)_s \\ \hline
\Box A, p\ \langle \Box A \mid \neg\Box\neg p, \neg\Box p\rangle & \\ \hline
\blacksquare A, A, p\ \langle \Box A \mid \neg\Box\neg p, \neg\Box p\rangle & \\ \hline
\blacksquare A, \neg\Box p, \neg\Box\neg p, p\ \langle \Box A \mid \neg\Box\neg p, \neg\Box p\rangle &
\end{array}$$

Some properties of the modal tableau system $\mathcal{C}\mathbf{S4+}$ are given in the following.

Definition 3.1 *The modal tableau system $\mathcal{C}\mathbf{S4+}$ has the* **analytical superformula property** *iff to every finite set Γ we can assign, a priori, a finite set $\Gamma^*_{\mathcal{C}\mathbf{S4+}}$ such that $\Gamma^*_{\mathcal{C}\mathbf{S4+}}$ contains all formulas that may appear in any $\mathcal{C}\mathbf{S4+}$-tableau for Γ.*

Fact 2 The modal tableau system $\mathcal{C}\mathbf{S4+}$ has the analytical superformula property, that is, for any finite set of formulas Γ, we can put $\Gamma^*_{\mathcal{C}\mathbf{S4+}} = Sub(\Gamma) \cup \neg Sub(\Gamma) \cup \blacksquare Sub_{\Box}(\Gamma) \cup \{f\}$, where $Sub(\Gamma)$ denotes the set of all subformulas of all $A \in \Gamma$, $\neg Sub(\Gamma) = \{\neg B | B \in Sub(\Gamma)\}$ and $\blacksquare Sub_{\Box}(\Gamma) = \{\blacksquare B | \Box B \in Sub(\Gamma)\}$.

Definition 3.2 *A tableau rule* (r) *is* **invertible** *in* $\mathcal{C}$**S4+** *iff if there is a closed* $\mathcal{C}$**S4+***-tableau for an instance of the numerator* N *of* (r) *then there are closed* $\mathcal{C}$**S4+***-tableaux for appropriate instances of the denominators* D_i *of* (r).

Lemma 3.3 *Each static rules of* $\mathcal{C}$**S4+** *is invertible in* $\mathcal{C}$**S4+**.

Proof) Similarly to the proof in [4]. □

We say that $\mathcal{C}$**S4+** is **sound with respect to S4-frames** if $\Gamma \vdash_{\mathcal{C}\mathbf{S4+}} A \Rightarrow \Gamma \models_{\mathbf{S4}} A$. In other words, if there is a closed $\mathcal{C}$**S4+**-tableau for $\Gamma \cup \{\neg A\}$ then any **S4**-model that makes Γ true at possible world w must make A true at w. We say that $\mathcal{C}$**S4+** is **complete with respect to S4-frames** if $\Gamma \models_{\mathbf{S4}} A \Rightarrow \Gamma \vdash_{\mathcal{C}\mathbf{S4+}} A$. In other words, if every **S4**-model that makes Γ true at possible world w also makes A at w, then some $\mathcal{C}$**S4+**-tableau for $\Gamma \cup \{\neg A\}$ must close.

3.2 Soundness of $\mathcal{C}$S4+

We show soundness of the modal tableau system $\mathcal{C}$**S4+**. Before that, we define the semantics of $\blacksquare A$ in Kripke frames as follows: Let $(W, R, \models)$ be a Kripke model.

$$w \models \blacksquare A \quad \textit{iff} \quad \textit{for all } w' \in W,\ wRw' \textit{ implies } w' \models A, \textit{ that is, } w \models \Box A.$$

Theorem 3.4 (soundness) *The modal tableau system* $\mathcal{C}$**S4+** *is sound with respect to* **S4***-frames.*

Proof) By definition of soundness, we show that if there is a closed $\mathcal{C}$**S4+**-tableau for $\Gamma \cup \{\neg A\}$ then $\Gamma \cup \{\neg A\}$ is **S4**-unsatisfiable. We can show this statement by induction on the structure of this $\mathcal{C}$**S4+**-tableau. Thus, $\Gamma \models_{\mathbf{S4}} A$ by Proposition 2.5. □

3.3 Completeness of $\mathcal{C}$S4+ via model graphs

Our completeness proof is based on Goré's [4] as follows: By definition, $\mathcal{C}$**S4+** is complete with respect to **S4**-frames if $\Gamma \models_{\mathbf{S4}} A \Rightarrow \Gamma \vdash_{\mathcal{C}\mathbf{S4+}} A$. We prove the contrapositive. That is, we assume that there is no $\mathcal{C}$**S4+**-tableau for $\Gamma \cup \{\neg A\}$. Then we pick and choose sets with certain special property from possibly different open tableaux for $\Gamma \cup \{\neg A\}$. We use them as possible worlds to construct an **S4**-model M for $\Gamma \cup \{\neg A\}$. The model M is deliberately constructed so as to contain a possible world w_0 such that $w_0 \models \Gamma \cup \{\neg A\}$. Hence we demonstrate by construction that $\Gamma \not\models_{\mathbf{S4}} A$. That is, proving completeness boils down to the following: if there is no closed $\mathcal{C}$**S4+**-tableau for $\Gamma \cup \{\neg A\}$ then there is an **S4**-model for $\Gamma \cup \{\neg A\}$ on an **S4**-frame (W, R).

As we have stated above, we shall associate sets of formulas with possible worlds. In our completeness proof of $\mathcal{C}$**S4+**, the difference with Goré's method is that we adopt a set of formulas with history $\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle$ as a possible world w. In order to prove completeness of $\mathcal{C}$**S4+** we need some technical machinery.

Definition 3.5 *A finite set* Γ *is* **closed with respect to a tableau rule** *if whenever (an instance of) the numerator of the rule is in* Γ*, so is (a corresponding instance of) at least one of the denominator of the rule. A single exception is the rule* $(T+)$*, where* Γ *is said to be closed with respect to* $(T+)$*, if both* $\blacksquare A$ *and* A *are in* Γ *when* $\Box A$ *is its principal formula. In this case, we do not require that* $\Box A$ *is in* Γ.

Definition 3.6 *A finite set* Γ *is* $\mathcal{C}$**S4+-saturated** *if it is* $\mathcal{C}$**S4+***-consistent and closed with respect to every static rule of* $\mathcal{C}$**S4+**.

Lemma 3.7 *For each finite* $\mathcal{C}$**S4+***-consistent* Γ *there is an effective procedure for constructing a finite* $\mathcal{C}$**S4+***-saturated* Γ^s *with* $\Gamma \subseteq \Gamma^s \subseteq \Gamma^*_{\mathcal{C}\mathbf{S4+}}$.

Proof) Similarly to the proof in [4]. □

Definition 3.8 (model graph for S4) *A* **model graph** *for a finite fixed set of formulas* Γ *is a finite* **S4***-frame* (W, R) *such that all* $w_i (= \Pi_i\langle\Box\Sigma_i \mid \neg\Box\Omega_i\rangle) \in W$ *and every* Π_i *is* $\mathcal{C}$**S4+***-saturated set with* $\Pi_i \subseteq \Gamma^*_{\mathcal{C}\mathbf{S4+}}$ *and*

1. $\Gamma \subseteq \Pi_0$ *for some* $w_0 \in W$,

2. *if* $\neg\Box A \in \Pi_i$ *then there exists some* $w_j \in W$ *with* $w_i R w_j$ *and* $\neg A \in \Pi_j$,

3. *if* $w_i R w_j$ *and* $\blacksquare A \in \Pi_i$ *then* $A \in \Pi_j$.

Definition 3.9 *Let* $w_i = \Pi_i\langle\Box\Sigma_i \mid \neg\Box\Omega_i\rangle$ *and* $w_j = \Pi_j\langle\Box\Sigma_j \mid \neg\Box\Omega_j\rangle$. *We say that two possible worlds* w_i *and* w_j *are equivalent if* $\Pi_i = \Pi_j$, $\Box\Sigma_i = \Box\Sigma_j$ *and* $\neg\Box\Omega_i = \neg\Box\Omega_j$.

Lemma 3.10 (satisfiability lemma) *If* (W, R) *is a model graph for a finite set* Γ *then there exists an* **S4***-model for* Γ.

Proof) Let Π be $\mathcal{C}$**S4**+-consistent set and $w = \Pi^s\langle\Box\Sigma \mid \neg\Box\Omega\rangle \in W$. For any $p \in \mathcal{P}$, we define a valuation as follows: $w \models p$ *iff* $p \in \Pi^s$. Below, similarly to the proof in [4]. $\Box$

Now, we give the sketch of completeness proof of $\mathcal{C}$**S4**+. As we have stated previously, proving completeness boils down to the following: if there is no closed $\mathcal{C}$**S4**+-tableau for $\Gamma \cup \{\neg A\}$ then there is an **S4**-model for $\Gamma \cup \{\neg A\}$ on an **S4**-frame. Our proof is as follows:

1. suppose $\Gamma \not\vdash_{\mathcal{C}\mathbf{S4}+} A$,
2. create a $\mathcal{C}$**S4**+-saturated $(\Gamma\cup\{\neg A\})^s$ from $\Gamma\cup\{\neg A\}$ (By Lemma 3.7) and put $w_0 = (\Gamma\cup\{\neg A\})^s\langle\epsilon \mid \epsilon\rangle$,
3. construct a finite model graph for $\Gamma \cup \{\neg A\}$ (By Theorem 3.13),
4. construct an **S4**-model for $\Gamma \cup \{\neg A\}$ (By Lemma 3.10),
5. Thus, $\Gamma \cup \{\neg A\}$ is **S4**-satisfiable,
6. $\Gamma \not\models_{\mathbf{S4}} A$ (By Proposition 2.5).

Remark 3 Let $\Gamma = \{p_1, \cdots, p_n, \neg q_1, \cdots, \neg q_m, \neg\Box A_1, \cdots, \neg\Box A_l\}$ and $(\Gamma\cup\blacksquare\Delta)\langle\Box\Sigma \mid \neg\Box\Omega\rangle$ be any node of $\mathcal{C}$**S4**+-tableaux. For every $(\Gamma\cup\blacksquare\Delta)\langle\Box\Sigma \mid \neg\Box\Omega\rangle$, $\Box\Sigma \subseteq \Box\Delta$ holds.

Proof) Clear from how to update history by applications of rules of $\mathcal{C}$**S4**+. $\Box$

Definition 3.11 *Let* Π *be any* $\mathcal{C}$**S4**+*-consistent set,* $\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle$ *be a node of* $\mathcal{C}$**S4**+*-tableaux and* $w = \Pi^s\langle\Box\Sigma \mid \neg\Box\Omega\rangle$. *The set* $\blacksquare\Delta \cup \{\neg\Box A_i\} \subseteq \Pi^s$ *is called a* **thinned set** *of* w*, where* $\blacksquare\Delta$ *is the set of all the formulas of the form* $\blacksquare A$ *in* Π^s *and* $\{\neg\Box A_i | 1 \leq i \leq l\}$ *is the set of all the formulas of the form* $\neg\Box A$ *in* Π^s. *The set* $\Box\Delta \cup \{\neg A_i\}$*, which can be obtained by a single application of* $(S4+)_t$ *or* $(S4+)_s$ *when either of them is applicable to* $(\blacksquare\Delta \cup \{\neg\Box A_i\})\langle\Box\Sigma \mid \neg\Box\Omega\rangle$*, is called a* **presaturated set** *from* w.

Note that each thinned set is stored on history when a presaturated set is obtained from it. We can see the following lemma.

Lemma 3.12 *Let* Π *be any* $\mathcal{C}$**S4**+*-consistent set,* $\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle$ *be a node of* $\mathcal{C}$**S4**+*-tableaux and* $w = \Pi^s\langle\Box\Sigma \mid \neg\Box\Omega\rangle$. *In addition, let* $\blacksquare\Delta \cup \{\neg\Box A_i\}$ *be a thinned set of* w*, where* $1 \leq i \leq l$. *If* $\Box\Sigma = \Box\Delta$ *and* $\neg\Box A_i \in \neg\Box\Omega$ *for some* $1 \leq i \leq l$ *then the presaturated set* $\Box\Delta \cup \{\neg A_i\}$ *from* w *have been already obtained.*

Proof) Since a thinned set is stored on history by a single application of $(S4+)_t$ or $(S4+)_s$, we can see whether a presaturated set have been already obtained or not. $\Box$

Theorem 3.13 (completeness) *If* Π *is a finite set of formulas and* $\mathcal{C}$**S4**+*-consistent then there is an* **S4***-model for* Π *on a finite* **S4***-frame.*

Proof) We give a way to construct a finite model graph (W, R) for Π. The first step is to create a $\mathcal{C}$**S4**+-saturated Π^s from Π with $\Pi \subseteq \Pi^s \subseteq \Pi^*_{\mathcal{C}\mathbf{S4}+}$. We can create Π^s by Lemma 3.7. Put $w_0 = \Pi_0\langle\epsilon \mid \epsilon\rangle$, where $\Pi_0 = \Pi^s$, corresponding to a possible world of W. We construct a model graph for Π based on w_0 as follows:
1) If no formula of the form $\neg\Box A$ occurs in Π_0 then $(W, R) = (\{w_0\}, \{(w_0, w_0)\})$ is the desired model graph for Π since this is **S4**-frame and satisfies conditions of model graph.

2) Otherwise, let $A_1, A_2, \cdots, A_m$ be all formulas such that $\neg\Box A_i \in \Pi_0$ and $\neg A_i \notin \Pi_0$ for each $1 \le i \le m$. Since Π_0 is $\mathcal{C}\mathbf{S4}$+-consistent, we can see that the thinned set of w_0, that is, $\blacksquare\Delta \cup \{\neg\Box A_i\} \subseteq \Pi_0$ also must be $\mathcal{C}\mathbf{S4}$+-consistent for each $1 \le i \le m$. By applying $(S4+)_t$ to $(\blacksquare\Delta \cup \{\neg\Box A_i\})\langle\epsilon \mid \epsilon\rangle$, and we can obtain $(\Box\Delta \cup \{\neg A_i\})\langle\Box\Delta \mid \neg\Box A_i\rangle$ (note that if $\blacksquare\Delta$ is empty then we apply $(S4+)_s$). Each presaturated set $\Box\Delta \cup \{\neg A_i\}$ from w_0 must be also $\mathcal{C}\mathbf{S4}$+-consistent. Then, put $w_0 R w_i$, where $w_i = (\Box\Delta \cup \{\neg A_i\})^s\langle\Box\Delta \mid \neg\Box A_i\rangle$.

Next, we give how to construct the successors of any $w_k (= \Pi_k\langle\Box\Sigma_k \mid \neg\Box\Omega_k\rangle)$ $(k \neq 0)$. Let $\blacksquare\Delta_k \cup \{\neg\Box A_j\}$ $(1 \le j \le n)$ be the thinned sets of w_k. Here, we have three cases by Lemma 3.3.

i) if $\Box\Sigma_k \subsetneq \Box\Delta_k$, apply $(S4+)_t$ to $(\blacksquare\Delta_k \cup \{\neg\Box A_j\})\langle\Box\Sigma_k \mid \neg\Box\Omega_k\rangle$ and we can obtain $(\Box\Delta_k \cup \{\neg A_j\})\langle\Box\Delta_k \mid \neg\Box A_j\rangle$. Then, put $w_k R w_{kj}$, where $w_{kj} = (\Box\Delta_k \cup \{\neg A_j\})^s\langle\Box\Delta_k \mid \neg\Box A_j\rangle$.

ii) if $\Box\Sigma_k = \Box\Delta_k$ and $\neg\Box A_j \notin \neg\Box\Omega_k$, apply $(S4+)_s$ to $(\blacksquare\Delta_k \cup \{\neg\Box A_j\})\langle\Box\Sigma_k \mid \neg\Box\Omega_k\rangle$ and we can obtain $(\Box\Delta_k \cup \{\neg A_j\})\langle\Box\Sigma_k \mid \neg\Box A_j, \neg\Box\Omega_k\rangle$. Then, put $w_k R w_{kj}$, where $w_{kj} = (\Box\Delta_k \cup \{\neg A_j\})^s\langle\Box\Sigma_k \mid \neg\Box A_j, \neg\Box\Omega_k\rangle$.

iii) if $\Box\Sigma_k = \Box\Delta_k$ and $\neg\Box A_j \in \neg\Box\Omega_k$ then, from Lemma 3.12, this means that there exists a certain possible world generated from $\Box\Sigma_k \cup \{\neg A_j\}$ as the predecessor of w_k. We can write the possible world as $w_h = (\Box\Sigma_k \cup \{\neg A_j\})^s\langle\Box\Sigma_k \mid \neg\Box\Omega'\rangle$, where $\neg\Box A_j \in \neg\Box\Omega'$. Put $w_k R w_h$. (We note that w_h is determined uniquely from how to construct successors.)

Since $\mathcal{C}\mathbf{S4}$+ has the analytical superformula property, successors are not infinitely generated. So, this construction terminates.

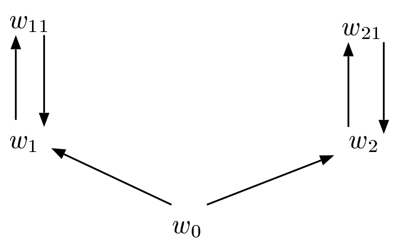
Finally, let W consist of all the nodes created in this process and take R^* as the reflexive and transitive closure of R. Then W is finite, and (W, R^*) is a model graph for Π from how to construct (W, R^*). Thus, by Lemma 3.10, we can obtain an $\mathbf{S4}$-model for Π with the root w_0. □

In the completeness proof, it is obvious that clusters are generated by constructing model graph and the clusters form a tree. As we see, our completeness proof of $\mathcal{C}\mathbf{S4}$+ actually gives the construction of counter-models for **S4** when the proof-search fails. In addition, **S4**-models constructed by our completeness proof are finite. It means that our completeness proof constructively shows that **S4** has the finite model property. Thus, we can say that $\mathcal{C}\mathbf{S4}$+ is not only a proof-search procedure for **S4** but also a procedure of construction of counter-models for **S4** when the proof-search fails.

3.4 Construction of counter-models

As we have stated, in fact, our completeness proof gives a procedure of construction of counter-models. In our method, a counter-model is constructed from a $\mathcal{C}\mathbf{S4}$+-consistent set semantically. On the other hand, Mouri also gives the construction of counter-models for **S4** based on sequent system. In his method, a counter-model is constructed from a failed proof figure proof-theoretically. Now, we give counter-models for $\neg\Box\neg(\Box p \vee \Box\neg p)$ by our method and Mouri's one. Let A denote $\neg(\Box p \vee \Box\neg p)$. The accessibility relation R is denoted by the arrows and the reflexive relations are omitted in the following.

Our method A counter-model for $\neg\Box A$ by our method is given in Figure 2.



$w_0 = \{\neg\neg\Box A, \blacksquare A, A, \neg\Box p, \neg\Box\neg p\}\langle\epsilon \mid \epsilon\rangle$
$w_1 = \{\blacksquare A, A, \neg\Box p, \neg\Box\neg p, \neg p\}\langle\Box A \mid \neg\Box p\rangle$
$w_{11} = \{\blacksquare A, A, \neg\neg p, p, \neg\Box p, \neg\Box\neg p\}\langle\Box A \mid \neg\Box\neg p, \neg\Box p\rangle$
$w_2 = \{\blacksquare A, A, \neg\neg p, p, \neg\Box p, \neg\Box\neg p\}\langle\Box A \mid \neg\Box\neg p\rangle$
$w_{21} = \{\blacksquare A, A, \neg\Box p, \neg\Box\neg p, \neg p\}\langle\Box A \mid \neg\Box p, \neg\Box\neg p\rangle$
$w_{11} \models p$, $w_2 \models p$

Figure 2: A counter-model for $\neg\Box\neg(\Box p \vee \Box\neg p)$ by our method

Mouri's method Next, we give a counter-model for $\neg\Box A$ by Mouri's method. The following is a failed proof figure for $\neg\Box A$ in Mouri's system.

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\hline\hline
\blacksquare A \to p, \Box p, \Box\neg p \ \langle \Box A \mid \Box p, \Box\neg p\rangle \\ \hline
\blacksquare A \to p, \Box p \vee \Box\neg p \ \langle \Box A \mid \Box p, \Box\neg p\rangle \\ \hline
\blacksquare A, A \to p \ \langle \Box A \mid \Box p, \Box\neg p\rangle \\ \hline
\Box A \to p \ \langle \Box A \mid \Box p, \Box\neg p\rangle
\end{array}
&
\begin{array}{c}
\hline\hline
\blacksquare A, p \to \Box p, \Box\neg p \ \langle \Box A \mid \Box p, \Box\neg p\rangle \\ \hline
\blacksquare A, p \to \Box p \vee \Box\neg p \ \langle \Box A \mid \Box p, \Box\neg p\rangle \\ \hline
\blacksquare A, A, p \to \ \langle \Box A \mid \Box p, \Box\neg p\rangle \\ \hline
\Box A, p \to \ \langle \Box A \mid \Box p, \Box\neg p\rangle \\ \hline
\Box A \to \neg p \ \langle \Box A \mid \Box p, \Box\neg p\rangle
\end{array}
\end{array} \\ \hline\hline
\blacksquare A \to \Box p, \Box\neg p \ \langle \Box A \mid \epsilon\rangle \\ \hline
\blacksquare A \to \Box p \vee \Box\neg p \ \langle \Box A \mid \epsilon\rangle \\ \hline
\blacksquare A, A \to \ \langle \Box A \mid \epsilon\rangle \\ \hline
\Box A \to \ \langle \epsilon \mid \epsilon\rangle \\ \hline
\to \neg\Box A \ \langle \epsilon \mid \epsilon\rangle
\end{array}
$$

Mouri's method is as follows: each part between double lines denotes a possible world. If the number of formulas of the form $\Box B$ of the left hand side of the top sequent in the upper part u is more than that of the left hand side of the top sequent in the lower part w, put wRu. Otherwise, put wRu and uRw. Repeat this process. Finally, take the reflexive and transitive closure of R.

A counter-model for $\neg\Box A$ by Mouri's method is given in Figure 3.

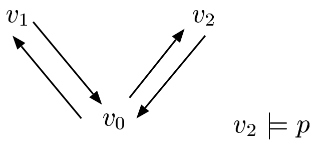


Figure 3: A counter-model for $\neg\Box\neg(\Box p \vee \Box\neg p)$ by Mouri's method

3.5 Termination of $\mathcal{C}$S4+

We claim that the modal tableau system $\mathcal{C}\mathbf{S4}+$ is loop-free, that is, same node never appear on each branch from the root to any node in every $\mathcal{C}\mathbf{S4}+$-tableau for any finite set Γ. In fact, we can show that applications of tableau rules of $\mathcal{C}\mathbf{S4}+$ never continue infinitely in the following.

Theorem 3.14 *The tableau procedure for $\mathcal{C}\mathbf{S4}+$ terminates.*

Proof) We show that the proof-search for all $\mathcal{C}\mathbf{S4}+$-tableaux for any finite set Γ terminates. First, for a formula A, we define the degree $deg(A)$ of A inductively as follows:

$$
\begin{array}{l}
deg(f) = 1, \\
deg(p) = 1 \text{ for all } p \in \mathcal{P}, \\
deg(\neg A) = 1 + deg(A), \\
deg(A \wedge B) = 2 + deg(A) + deg(B), \\
deg(A \vee B) = 2 + deg(A) + deg(B), \\
deg(A \supset B) = 2 + deg(A) + deg(B), \\
deg(\Box A) = 3 \cdot deg(A), \\
deg(\blacksquare A) = deg(A).
\end{array}
$$

For a finite set $\Delta = \{A_1, \cdots, A_m\}$, where A_i and A_j are mutually distinct when $i \neq j$,

$$deg(\Delta) = \sum_{i=1}^{m} deg(A_i).$$

Next, for any node $\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle$ in any $\mathcal{C}\mathbf{S4}+$-tableau for Γ, we define the degree $Deg(\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle) \in \mathbf{N}^3$ of $\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle$ as follows:

$$Deg(\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle) = (|\Gamma^*_{\mathcal{C}\mathbf{S4}+}| - |\Box\Sigma|,\ |\Gamma^*_{\mathcal{C}\mathbf{S4}+}| - |\neg\Box\Omega|,\ deg(\Pi)).$$

We note that $|\Gamma^*_{\mathcal{C}\mathbf{S4+}}| - |\Box\Sigma|$, $|\Gamma^*_{\mathcal{C}\mathbf{S4+}}| - |\neg\Box\Omega| \geq 1$. In addition, let x_1, x_2, x_3, y_1, y_2 and y_3 be any natural numbers. We define a lexicographic order $\ll$ over triples of natural numbers as follows:

$$(x_1, x_2, x_3) \ll (y_1, y_2, y_3) \quad \textit{iff} \quad x_1 < y_1 \textit{ or } (x_1 = y_1 \textit{ and } x_2 < y_2) \textit{ or } (x_1 = y_1, x_2 = y_2 \textit{ and } x_3 < y_3).$$

With an application of $(S4+)_t$, $|\Gamma^*_{\mathcal{C}\mathbf{S4+}}| - |\Box\Sigma|$ strictly decreases, while with an application of $(S4+)_s$, $|\Gamma^*_{\mathcal{C}\mathbf{S4+}}| - |\neg\Box\Omega|$ strictly does. Although history is not updated with each application of a static rule of $\mathcal{C}\mathbf{S4+}$, $deg(\Pi)$ strictly decreases with it. Thus, for every $\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle$, $Deg(\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle)$ strictly decreases with each application of a rule of $\mathcal{C}\mathbf{S4+}$. □

3.6 Time complexity of $\mathcal{C}$S4+

By our proof of termination of $\mathcal{C}\mathbf{S4+}$, we can estimate the time complexity of $\mathcal{C}\mathbf{S4+}$, that is, the total number of applications of tableau rules in all $\mathcal{C}\mathbf{S4+}$-tableaux for any finite set Γ. Time complexity depends much on procedure. Time and space complexity on some logics have been discussed in [9, 7, 8].

First, for a formula A, we define the length $len(A)$ of A inductively as follows:

$$\begin{aligned}
&len(f) = 1,\\
&len(p) = 1 \text{ for all } p \in \mathcal{P},\\
&len(\neg A) = 1 + len(A),\\
&len(A \wedge B) = 1 + len(A) + len(B),\\
&len(A \vee B) = 1 + len(A) + len(B),\\
&len(A \supset B) = 1 + len(A) + len(B),\\
&len(\Box A) = 1 + len(A),\\
&len(\blacksquare A) = 1 + len(A).
\end{aligned}$$

For a finite set $\Delta = \{A_1, \cdots, A_m\}$, where A_i and A_j are mutually distinct when $i \neq j$,

$$len(\Delta) = \sum_{i=1}^{m} len(A_i).$$

Here, let $Sub_{mul}(A)$ denote the multiset of all subformulas of A and $Sub_{mul}(\Gamma)$ denote the multiset of all subformulas of all $A \in \Gamma$.

Lemma 3.15 *For any formula A, $|Sub(A)| \leq len(A)$. Hence, for any finite set Γ, $|Sub(\Gamma)| \leq len(\Gamma)$. In addition, $|Sub_{mul}(A)| = len(A)$. Hence, $|Sub_{mul}(\Gamma)| = len(\Gamma)$.*

Proof) By induction on the structure of A. □

Theorem 3.16 *Let $T_{\mathcal{C}\mathbf{S4+}}(\Gamma)$ denote the time complexity of $\mathcal{C}\mathbf{S4+}$ for any finite set Γ. Then, the upper bound of $T_{\mathcal{C}\mathbf{S4+}}(\Gamma)$ can be estimated at $(2l)^{9l^2+3l+1} \cdot (2l+1) - 4l^2$, where $l = len(\Gamma)$. In other words, for the input Γ with the length of l, $\mathcal{C}\mathbf{S4+}$ terminates in the steps of $(2l)^{9l^2+3l+1} \cdot (2l+1) - 4l^2$ at worst.*

Proof) Let $c = |\Gamma^*_{\mathcal{C}\mathbf{S4+}}|$. We can count the total number of applications of tableau rules in all $\mathcal{C}\mathbf{S4+}$-tableaux for Γ with the following measure:

$$Deg(\Pi\langle\Box\Sigma \mid \neg\Box\Omega\rangle) = (c - |\Box\Sigma|,\ c - |\neg\Box\Omega|,\ deg(\Pi)).$$

First, by fixing $deg(\Pi)$, we count the number of applications of transitional rules. At worst case, every element of $(c-|\Box\Sigma|, c-|\neg\Box\Omega|)$ decreases one by one. This means that $(c-|\Box\Sigma|, c-|\neg\Box\Omega|)$ runs over root point and the grid points of shadowed square as shown Figure 4. Here, in each application of transitional rules, we must choose a formula of the form $\neg\Box A$. Hence, there will be several possibilities of the choice of formulas of the form $\neg\Box A$ in each application of transitional rules (see Figure 5). Let b be the number of branches which each node has in the tree, that is, the number of formulas of the form $\neg\Box A$ to be chosen in each application of transitional rules. The height of the tree is $(c-1)c$. The total number of applications of transitional rules can be regarded as the total number of branches of the tree. Thus,

$$b \ + \ b^2 + \ \cdots + \ b^{(c-1)c} = \frac{b\{b^{(c-1)c} - 1\}}{b-1}$$

The total number of the nodes of the tree is:

$$1 \; + \; b \; + \; b^2 \; + \; \cdots + \; b^{(c-1)c} = 1 + \frac{b\{b^{(c-1)c} - 1\}}{b-1}$$

Let s be the number of applications of static rules which are applied before the next application of transitional rule. Note that each transitional rule becomes applicable only after some applications of static rules in $\mathcal{C}\mathbf{S4}+$. Then, the total number of applications of static rules is:

$$s \cdot \left(1 + \frac{b\{b^{(c-1)c} - 1\}}{b-1}\right)$$

Therefore, we can estimate the time complexity as follows:

$$\begin{aligned} T_{\mathcal{C}\mathbf{S4}+}(\Gamma) &\leq s \cdot \left(1 + \frac{b\{b^{(c-1)c} - 1\}}{b-1}\right) + \frac{b\{b^{(c-1)c} - 1\}}{b-1} \\ &= (s+1) \cdot \frac{b\{b^{(c-1)c} - 1\}}{b-1} + s \end{aligned}$$

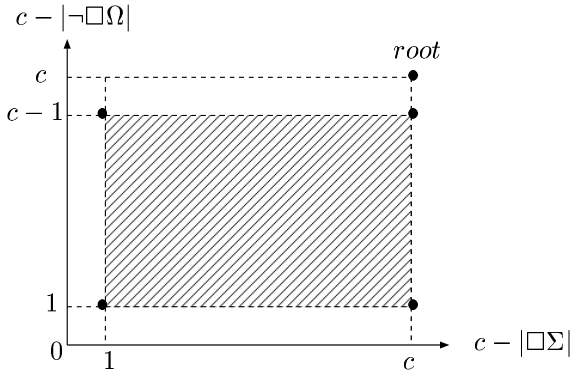


Figure 4: The total number of applications of transitional rules (1)

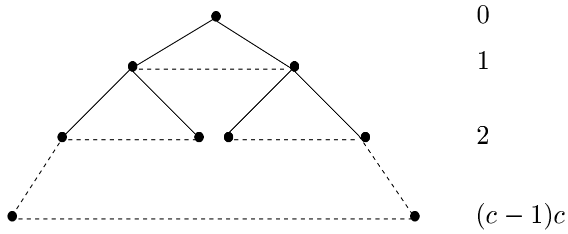


Figure 5: The total number of applications of transitional rules (2)

Now, it remains to estimate c, b and s in terms of the length of Γ. Let $l = len(\Gamma)$. From Lemma 3.15:

$$\begin{aligned} c &= |\Gamma^*_{\mathcal{C}\mathbf{S4}+}| \\ &= |Sub(\Gamma) \cup \neg Sub(\Gamma) \cup \blacksquare Sub_{\Box}(\Gamma) \cup \{f\}| \\ &\leq |Sub(\Gamma)| + |\neg Sub(\Gamma)| + |\blacksquare Sub_{\Box}(\Gamma)| + 1 \\ &\leq 3l + 1. \end{aligned}$$

The number of applications of static rules which are applied before the next application of transitional rule can be regarded as the number of principal formulas in $Sub(\Gamma) \cup \neg Sub(\Gamma)$. Here, note that formulas of the same form may become principal formula many times. So, we should consider that all the principal formulas are in $Sub_{mul}(\Gamma) \cup \neg Sub_{mul}(\Gamma)$. Moreover, formulas of the form $\neg\Box A$ are in $Sub(\Gamma) \cup \neg Sub(\Gamma)$. From Lemma 3.15:

$$\begin{array}{rcl} s & \leq & |Sub_{mul}(\Gamma) \cup \neg Sub_{mul}(\Gamma)| \\ & = & |Sub_{mul}(\Gamma)| + |\neg Sub_{mul}(\Gamma)| \\ & = & 2l, \\ b & \leq & |Sub(\Gamma) \cup \neg Sub(\Gamma)| \\ & \leq & |Sub(\Gamma)| + |\neg Sub(\Gamma)| \\ & \leq & 2l. \end{array}$$

Thus, it follows that:

$$\begin{array}{rcl} T_{\mathcal{C}\mathbf{S4+}}(\Gamma) & \leq & (s+1) \cdot \dfrac{b\{b^{(c-1)c} - 1\}}{b-1} + s \\ & \leq & (s+1) \cdot \Big(b\{b^{(c-1)c} - 1\}\Big) + s \\ & \leq & (2l)^{9l^2+3l+1} \cdot (2l+1) - 4l^2. \end{array} \qquad \Box$$

4 Conclusion

In this paper, we give an improved modal tableau system $\mathcal{C}$**S4+** and show that $\mathcal{C}$**S4+** is sound and complete with respect to **S4**-frames. As a result, the modal tableau system $\mathcal{C}$**S4+** gives not only a proof-search procedure for **S4** but also a procedure of construction of counter-models for **S4** when the proof-search fails. Our construction of counter-models is semantical, while Mouri's one is proof-theoretical.

Since $\mathcal{C}$**S4+** is loop-free, $\mathcal{C}$**S4+**-tableaux are also loop-free itself. It follows that $\mathcal{C}$**S4+** gives an efficient proof-search procedure, and therefore we can say that $\mathcal{C}$**S4+** is suitable for implementation of theorem prover for **S4**. Moreover, because $\mathcal{C}$**S4+** is loop-free, we can estimate the total number of applications of tableau rules in terms of the length of the input until $\mathcal{C}$**S4+** terminates. The time complexity is discussed explicitly neither in Heuerding, Seyfried and Zimmermann's system nor in Mouri's one. It would be possible to apply our method of estimation of time complexity to other systems.

Although $\mathcal{C}$**S4+** are equipped with the auxiliary model operator $\blacksquare$ and history as Mouri's system, we have shown that $\mathcal{C}$**S4+** is loop-free. In other words, we have improved Mouri's system into a loop-free system.

5 Further work

Further work is given in the following.

- **Proof-theoretical construction of counter-model for S4**
  Though counter-models are constructed semantically by our method, it would be possible to construct counter-models from open $\mathcal{C}$**S4+**-tableaux proof-theoretically by modifying tableau rules of $\mathcal{C}$**S4+**.

- **Implementation of $\mathcal{C}$S4+**
  Since the modal tableau system $\mathcal{C}$**S4+** is an efficient proof-search procedure for **S4**, it is suitable for implementation theorem prover for **S4**.

- **Loop-free systems for other logics**
  A loop-free system for **S4** has been given by making good use of history. Loop-free systems could be given by applying history to the systems for other logics similarly. The loop-free systems give us efficient proof-search procedures.

Acknowledgments

The author would like to express his thanks to the referees and M. Mouri for their helpful comments.

References

[1] M. C. Fitting, Tableau methods of proof for modal logics, *Norte Dame Journal of Formal Logic* 13, pp.237-247, 1972.

[2] M. C. Fitting, Proof of Methods for Modal and Intuitionistic Logics, D.Reidel Publishing Co., Dordrecht, 1983.

[3] R. Goré, Cut-free sequent and tableau systems for propositional Diodorean modal logics, *Studia Logica* 53, pp.433-457, 1994.

[4] R. Goré, Tableau Methods for Modal and Temporal Logics, in Handbook of Tableau Methods, eds. by Marcello D'Agostino, Dov M.Gabby, Reiner Hähnle, Joachim Posegga, Kluwer Academic Publishers, 1999.

[5] A. Heuerding, M. Seyfried and H. Zimmermann, Efficient Loop-Check for Backward Proof Search in Some Non-Classical Propositional Logics, in *Proceedings of the 5th Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, LNCS 1071, pp. 210-225. 1996.

[6] J. Hudelmaier, On a Contraction Free Sequent Calculus for the Modal Logic $S4$, in *Proceedings of the 3rd Workshop on Theorem Proving with Analytic Tableaux and Related Methods.* Technical Report TR-94/5, Department of Computing, Imperial College of Science, Technology and Medicine, London, 1994.

[7] J. Hudelmaier, Improved Decision Procedures for the Modal Logics K and T and $S4$, in *Proceeding of the 9th International Workshop, Computer Science Logic '95*, LNCS 1092, pp. 320-334, 1996.

[8] M. Kracht, Notes on the Space Requirements for Checking Satisfiability in Modal Logics, in *Proceeding of the 4th International Workshop, Advances in Modal Logic 2002*, pp. 205-221.

[9] , R. E. Ladner, The Computational Complexity of Provability in Systems of Modal Propositional Logic, *SIAM Journal on Computing*, 6(3):467-480, 1977.

[10] T. Matsumoto and H. Ono, Theorem Prover for Modal Logics with Tableau Method, in *Proceedings of International Workshop, Rewriting in Proof and Computation '01*, pp. 42-59, 2001.

[11] M. Mouri, Theorem Provers with Countermodels and xpe, *Bulletin of the Section of Logic*, 30(2):79-86, 2001

[12] M. Mouri, An Efficient Construction of Counter-models for Modal Logic S4, in *Proceedings of International Workshop, Rewriting in Proof and Computation '01*, pp. 247-256, 2001.

[13] H. Ono, Proof-theoretical methods in nonclassical logic – an introduction, Theories of Types and Proofs, eds. by M. Takahashi et al, MSJ Memoir 2, *Mathematical Society of Japan*, pp. 207-254, 1998.

[14] N. Rescher and A. Urquhart, Temporal Logic, Springer-Verlag, 1971.

[15] R. M. Smullyan, First-Order Logic, Springer-Verlag, 1968, Revised edition, Dover Press, 1994.